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Using factorization to solve soliton equations

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Abstract. It was shown in our previous paper that each equation in a soliton hierarchy can be factorized into two commuting x- and t_n-finite-dimensional integrable Hamiltonian systems (FDIHSs). The separation variables for these FDIHSs are constructed by using their Lax representation. By means of the factorization and the separability of the FDIHSs we obtain the Jacobi inversion problem, which is solvable in terms of Riemann theta functions, for soliton equations. This provides a method analogous to the separation of variables for solving soliton equations.

The separation of variables is one of the most universal methods of solving a completely integrable Hamiltonian system. For classical integrable systems subject to an inverse scattering transformation the standard construction of the action–angle variables using the poles of the Baker–Akheizer function is equivalent to separation of variables [1]. The finite-gap solutions of the soliton equation are constructed by means of the separation of variables of the stationary soliton equation [2, 3].

It was shown in [4, 5] that each equation in a soliton hierarchy can be factorized into two commuting x- and t_n-finite-dimensional integrable Hamiltonian systems (FDIHSs). The Lax representation for these FDIHSs can always be deduced from the adjoint representation of the auxiliary linear problem for the soliton equations [6]. Recently, much interest has developed in the study of the separation of variables for FDIHSs with a Lax representation [1, 7–12]. By using the factorization of soliton equations and the separation of variables for the FDIHSs we obtain the Jacobi inversion problem, which can be solved in terms of Riemann theta functions, for soliton equations. This provides a method analogous to the separation of variables for solving soliton equations. We illustrate the method by the Jaulent–Miodek hierarchy.

The Jaulent–Miodek (JM) eigenvalue problem [13] reads

$$\psi_x = U(u,\lambda)\psi \qquad U(u,\lambda) = \begin{pmatrix} 0 & 1 \\ -\lambda^2+\lambda q+r & 0 \end{pmatrix} \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad u = \begin{pmatrix} q \\ r \end{pmatrix}. \tag{1}$$

The adjoint representation of (1) [3, 14] is

$$V_x = [U, V] \equiv UV - VU. \tag{2}$$

Set

$$V = \sum_{m=0}^{\infty} V_m \lambda^{-m} \qquad V_m = \begin{pmatrix} a_m & b_m \\ c_m & -a_m \end{pmatrix}. \tag{3}$$

Equation (2) yields

$$a_0 = a_1 = a_2 = b_0 = b_1 = 0 \qquad b_2 = -1 \qquad b_3 = -\tfrac{1}{2}q$$
$$c_0 = 1 \qquad c_1 = -\tfrac{1}{2}q \qquad \begin{pmatrix} b_{m+2} \\ b_{m+1} \end{pmatrix} = L \begin{pmatrix} b_{m+1} \\ b_m \end{pmatrix} \qquad m = 1, 2, \ldots \tag{4a}$$
$$a_m = -\tfrac{1}{2}b_{m,x} \qquad c_m = a_{m,x} - b_{m+2} + q b_{m+1} + r b_m \qquad m = 1, 2, \ldots$$
$$L = \begin{pmatrix} q - \frac{1}{2}\partial_x^{-1} q_x & r - \frac{1}{2}\partial_x^{-1} r_x - \frac{1}{4}\partial_x^2 \\ 1 & 0 \end{pmatrix}. \tag{4b}$$

Take

$$N^{(n)}(u, \lambda) = \sum_{m=0}^{n} V_m \lambda^{n-m} + \triangle_n \qquad \triangle_n\,(u, \lambda) = \begin{pmatrix} 0 & 0 \\ \lambda b_{n+1} + b_{n+2} - q b_{n+1} & 0 \end{pmatrix} \tag{5}$$

and let

$$\psi_{t_n} = N^{(n)}(u, \lambda)\psi = \left(\sum_{m=0}^{n} V_m \lambda^{n-m} + \triangle_n \right) \psi. \tag{6}$$

Then the compatibility condition of (1) and (6) gives rise to the zero-curvature equation $U_{t_n} - N_x^{(n)} - [U, N^{(n)}] = 0$, which leads to the JM hierarchy:

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J \begin{pmatrix} b_{n+2} \\ b_{n+1} \end{pmatrix} = J \frac{\delta H_{n+1}}{\delta u} \tag{7}$$

$$J = \begin{pmatrix} 0 & 2\partial_x \\ 2\partial_x & -q_x - 2q\partial_x \end{pmatrix} \qquad H_1 = -q \qquad H_m = \frac{1}{m-1}(2b_{m+2} - q b_{m+1}). \tag{8}$$

Also we have

$$\frac{\delta \lambda}{\delta u} = \frac{1}{2} \begin{pmatrix} \lambda \psi_1^2 \\ \psi_1^2 \end{pmatrix}. \tag{9}$$

Furthermore V satisfies the adjoint representation of (6) [3]:

$$V_{t_n} = [N^{(n)}, V] \qquad n = 1, 2, \ldots . \tag{10}$$

The x-constrained flow of (7) consists of replicas of (1) for N distinct λ_j and of a restriction of the variational derivatives for the conserved quantities H_k (for any fixed k) and λ_j [4, 5]:

$$\Psi_{1x} = \Psi_2 \qquad \Psi_{2x} = -\Lambda^2 \Psi_1 + q \Lambda \Psi_1 + r \Psi_1 \tag{11a}$$

$$\frac{\delta H_{k+1}}{\delta u} + \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \begin{pmatrix} b_{k+2} \\ b_{k+1} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \langle \Lambda \Psi_1, \Psi_1 \rangle \\ \langle \Psi_1, \Psi_1 \rangle \end{pmatrix} = 0. \tag{11b}$$

Hereafter we denote the inner product in $\mathbb{R}^N$ by $\langle \cdot, \cdot \rangle$ and $\Psi_i = (\psi_{i1}, \cdots, \psi_{iN})^{\mathrm{T}}$, $i = 1, 2$, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$. The system (11) is invariant under all flows of (7). By introducing the Jacobi–Ostrogradsky coordinates, equations (11) can be transformed into a x-FDIHS.

The t_n-constrained flow of (7) consists of replicas of (6) for N distinct λ_j and of equation (7):

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix} t_n = N^{(n)}(u, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix} \qquad j = 1, \ldots, N \tag{12a}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J \begin{pmatrix} b_{n+2} \\ b_{n+1} \end{pmatrix}. \tag{12b}$$

Under equations (11) and the Jacobi–Ostrogradsky coordinates introduced above, equations (12) can be transformed into a t_n-FDIHS. If (q, r, Ψ_1, Ψ_2) satisfies these two commuting x- and t_n-FDIHS, then (q, r) solve the soliton equation (7), i.e. the x- and t_n-dependence of (7) are factorized by these two x- and t_n-FDIHS. Therefore, some kind of solution, such as a finite-gap solution, for equation (7) can be obtained through solving the two commuting x- and t_n-FDIHS obtained from (11) and (12). We shall find the Jacobi inversion problem for these x- and t_n-FDIHS later, and combine them to give the Jacobi inversion problem for equation (7), which is solvable in terms of the Riemann theta function.

The Lax representation for (11), which can be deduced from the adjoint representation (2), is of the form [6]

$$M_x^{(k)} = [U, M^{(k)}] \tag{13}$$

where

$$M^{(k)} = \sum_{m=0}^{k} V_m \lambda^{k-m} + N_0 \qquad N_0 = \frac{1}{2} \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \psi_{1j}\psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 & -\psi_{1j}\psi_{2j} \end{pmatrix}. \tag{14}$$

The Lax representation for (12), which can be deduced from the adjoint representation (10), is given by

$$M_{t_n}^{(k)} = [N^{(n)}, M^{(k)}] \tag{15}$$

which shares the same Lax matrix $M^{(k)}$ with (13).

When $k = 2$, equation (11*b*) reads

$$q = \langle \Psi_1, \Psi_1 \rangle \qquad r = \langle \Lambda\Psi_1, \Psi_1 \rangle - \tfrac{3}{4}\langle \Psi_1, \Psi_1 \rangle^2 \tag{16}$$

and equation (11*a*) becomes

$$\Psi_{1x} = \frac{\partial \widetilde{H}_0}{\partial \Psi_2} \qquad \Psi_{2x} = -\frac{\partial \widetilde{H}_0}{\partial \Psi_1} \tag{17a}$$

$$\widetilde{H}_0 = \tfrac{1}{2}\langle \Psi_2, \Psi_2 \rangle + \tfrac{1}{2}\langle \Lambda^2\Psi_1, \Psi_1 \rangle - \tfrac{1}{2}\langle \Psi_1, \Psi_1 \rangle \langle \Lambda\Psi_1, \Psi_1 \rangle + \tfrac{1}{8}\langle \Psi_1, \Psi_1 \rangle^3. \tag{17b}$$

For $n = 3$, under (16) and (17) equation (12) becomes

$$\Psi_{1t_3} = \frac{\partial \widetilde{H}_3}{\partial \Psi_2} \qquad \Psi_{2t_3} = -\frac{\partial \widetilde{H}_3}{\partial \Psi_1} \tag{18a}$$

$$\begin{aligned} \widetilde{H}_3 = {} & -\tfrac{1}{2}\langle \Lambda\Psi_2, \Psi_2 \rangle - \tfrac{1}{2}\langle \Lambda^3\Psi_1, \Psi_1 \rangle + \tfrac{1}{4}\langle \Psi_1, \Psi_1 \rangle \langle \Lambda^2\Psi_1, \Psi_1 \rangle \\ & -\tfrac{1}{8}\langle \Psi_1, \Psi_1 \rangle^2 \langle \Lambda\Psi_1, \Psi_1 \rangle - \tfrac{1}{4}\langle \Psi_1, \Psi_1 \rangle \langle \Psi_2, \Psi_2 \rangle + \tfrac{1}{4}\langle \Lambda\Psi_1, \Psi_1 \rangle^2. \end{aligned} \tag{18b}$$

For $n = 4$, under (16) and (17) equation (12) becomes

$$\Psi_{1t_4} = \frac{\partial \widetilde{H}_4}{\partial \Psi_2} \qquad \Psi_{2t_4} = -\frac{\partial \widetilde{H}_4}{\partial \Psi_1} \tag{19a}$$

$$\begin{aligned} \widetilde{H}_4 = {} & -\tfrac{1}{2}\langle \Lambda^2\Psi_2, \Psi_2 \rangle - \tfrac{1}{2}\langle \Lambda^4\Psi_1, \Psi_1 \rangle + \tfrac{1}{4}\langle \Psi_1, \Psi_1 \rangle \langle \Lambda^3\Psi_1, \Psi_1 \rangle \\ & -\tfrac{1}{8}\langle \Psi_1, \Psi_1 \rangle^2 \langle \Lambda^2\Psi_1, \Psi_1 \rangle - \tfrac{1}{4}\langle \Psi_1, \Psi_1 \rangle \langle \Lambda\Psi_2, \Psi_2 \rangle \\ & -\tfrac{1}{4}\langle \Lambda\Psi_1, \Psi_1 \rangle \langle \Psi_2, \Psi_2 \rangle + \tfrac{1}{2}\langle \Psi_1, \Psi_2 \rangle \langle \Lambda\Psi_1, \Psi_2 \rangle \\ & +\tfrac{1}{4}\langle \Lambda\Psi_1, \Psi_1 \rangle \langle \Lambda^2\Psi_1, \Psi_1 \rangle. \end{aligned} \tag{19b}$$

Then the soliton equation (7) with $n = 3$ ($n = 4$) is factorized by (17) and (18) (equations (19)), i.e. if (Ψ_1, Ψ_2) satisfies two commuting FDIHS (17) and (18) (equations (19)) simultaneously, then (q, r) given by (16) solve the soliton equation (7) with $n = 3$ ($n = 4$).

The Lax matrix $M^{(2)}$ for equations (17), (18) and (19) is of the form

$$M^{(2)} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}$$

$$A(\lambda) = \frac{1}{2}\sum_{j=1}^{N} \frac{\psi_{1j}\psi_{2j}}{\lambda - \lambda_j} \qquad B(\lambda) = -1 - \frac{1}{2}\sum_{j=1}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j} \tag{20a}$$

$$C(\lambda) = \lambda^2 - \frac{1}{2}\langle\Psi_1, \Psi_1\rangle\lambda - \frac{1}{2}\langle\Lambda\Psi_1, \Psi_1\rangle + \frac{1}{4}\langle\Psi_1, \Psi_1\rangle^2 + \frac{1}{2}\sum_{j=1}^{N} \frac{\psi_{2j}^2}{\lambda - \lambda_j}. \tag{20b}$$

We introduce the separation variables $u_k, v_k, k = 1, \ldots, N$ by the zeros of $B(\lambda)$:

$$-B(\lambda) = 1 + \frac{1}{2}\sum_{j=1}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j} = \frac{R(\lambda)}{K(\lambda)} \tag{21a}$$

and

$$v_k = A(u_k) = \frac{1}{2}\sum_{j=1}^{N} \frac{\psi_{1j}\psi_{2j}}{u_k - \lambda_j} \qquad k = 1, \ldots, N \tag{21b}$$

where

$$K(\lambda) \equiv \prod_{j=1}^{N}(\lambda - \lambda_j) = \sum_{i=0}^{N} \alpha_i \lambda^{N-i} \qquad R(\lambda) \equiv \prod_{j=1}^{N}(\lambda - u_j)$$

$$\alpha_0 = 1 \qquad \alpha_1 = -\sum_{j=1}^{N} \lambda_j \qquad \alpha_2 = \sum_{j=1}^{N}\sum_{k=j+1}^{N} \lambda_j\lambda_k, \ldots .$$

It follows from equation (21a) that

$$\psi_{1j}^2 = 2\frac{R(\lambda_j)}{K'(\lambda_j)} \qquad j = 1, \ldots, N \tag{22}$$

where the prime denotes differentiation with respect to λ. Then from equation (22) we obtain

$$\sum_{j=1}^{N} \psi_{2j}\mathrm{d}\psi_{1j} = \sum_{j=1}^{N} v_k \mathrm{d}u_k \tag{23}$$

which implies that the coordinates u_k, v_k are canonically conjugate. From equation (21a) we find

$$q = \langle\Psi_1, \Psi_1\rangle = 2\beta_1 - 2\alpha_1 \tag{24a}$$

$$r = \langle\Lambda\Psi_1, \Psi_1\rangle - \tfrac{3}{4}\langle\Psi_1, \Psi_1\rangle^2 = 2\beta_2 - 2\alpha_2 + 4\beta_1\alpha_1 - 3\beta_1^2 - \alpha_1^2 \tag{24b}$$

where

$$\beta_1 = -\sum_{j=1}^{N} u_j \qquad \beta_2 = \sum_{j=1}^{N}\sum_{k=j+1}^{N} u_j u_k.$$

The equalities (13) and (15) indicate that $\frac{1}{2}\,\mathrm{Tr}(M^{(2)}(\lambda))^2 = A^2(\lambda) + B(\lambda)C(\lambda)$ is the generating function of integrals of motion for the system (17), (18) and (19). Let

$$A^2(\lambda) + B(\lambda)C(\lambda) = \frac{P(\lambda)}{K(\lambda)} \qquad P(\lambda) = \sum_{i=0}^{N+2} P_i \lambda^i \tag{25}$$

where $P_i, i = 0, 1, \ldots, N-1$, are the integrals of motion for (17), (18) and (19). By substituting (20) we find

$$P_{N+2} = -1 \qquad P_{N+1} = -\alpha_1 \qquad P_N = -\alpha_2$$

$$\widetilde{H}_0 = -P_{N-1} - \alpha_3 \qquad \widetilde{H}_3 = P_{N-2} - \alpha_1 P_{N-1} - \alpha_1\alpha_3 + \alpha_4 \tag{26a}$$

$$\widetilde{H}_4 = P_{N-3} - \alpha_1 P_{N-2} + (\alpha_1^2 - \alpha_2)P_{N-1} + \alpha_1^2\alpha_3 - \alpha_1\alpha_4 - \alpha_2\alpha_3 + \alpha_5, \ldots\,. \tag{26b}$$

In order to write the Hamilton–Jacobi equation from (25), we must reinterpret the P_i as integration constants and replace v_k by the partial derivatives $\partial S/\partial u_k$ of the generating function S of the canonical transformation [15]. Inserting $\lambda = u_k$, from (25) we find

$$v_k = \sqrt{\frac{P(u_k)}{K(u_k)}} \qquad k = 1, \ldots, N \tag{27}$$

which implies that the variables in the Hamilton–Jacobi equation are completely separable. S can be expressed in the separation form $S(u_1, \ldots, u_N) = \sum_{k=1}^{N} S_k(u_k)$. By replacing $v_k = \partial S_k/\partial u_k$ and interpreting the P_i as integration constants, equation (25) may be integrated to give the completely separated solution

$$S(u_1, \ldots, u_N) = \sum_{k=1}^{N} \int^{u_k} \sqrt{\frac{P(\lambda)}{K(\lambda)}}\, \mathrm{d}\lambda. \tag{28}$$

Obviously, defining the separation variables $u_k, k = 1, \ldots, N$ by the zeros of $B(\lambda)$ and $v_k, k = 1, \ldots, N$ by $v_k = A(u_k)$ ensures that the separation equations (27) can be deduced from the generating function of the integrals of motion (25).

The linearizing coordinates are then

$$Q_i \equiv \frac{\partial S}{\partial P_i} = \frac{1}{2} \sum_{k=1}^{N} \int^{u_k} \frac{\lambda^i}{\sqrt{P(\lambda)K(\lambda)}}\, \mathrm{d}\lambda \qquad i = 0, 1, \ldots, N-1. \tag{29}$$

This equality provides a map, called the Abel map, from the old coordinates $u_k, k = 1, \ldots, N$, which live on the Riemann surface, to new coordinates $Q_k, k = 0, 1, \ldots, N-1$ which live on its Jacobi variety. The linear flow induced by (17) is then given by (using equation (26*a*))

$$Q_i = c_i + \frac{\partial \widetilde{H}_0}{\partial P_i} x = c_i - x\delta_{i,N-1} \qquad i = 0, 1, \ldots, N-1. \tag{30}$$

The linear flow induced by (18) is of the form (using equation (26*a*))

$$Q_i = \bar{c}_i + \frac{\partial \widetilde{H}_3}{\partial P_i} t_3 = \bar{c}_i + [\delta_{i,N-2} - \alpha_1\delta_{i,N-1}]t_3 \qquad i = 0, 1, \ldots, N-1. \tag{31}$$

Combining equations (29), (30) and (31) gives rise to the Jacobi inversion problem for the soliton equation (7) with $n = 3$:

$$\frac{1}{2} \sum_{k=1}^{N} \int^{u_k} \frac{\lambda^i}{\sqrt{P(\lambda)K(\lambda)}}\, \mathrm{d}\lambda = c_i - \delta_{i,N-1}(x + \alpha_1 t_3) + \delta_{i,N-2} t_3 \qquad i = 0, 1, \ldots, N-1. \tag{32}$$

The linear flow induced by (19) is given by (using equation (26*b*))

$$Q_i = \bar{c}_i + \frac{\partial \widetilde{H}_4}{\partial P_i} t_4 = \bar{c}_i + [\delta_{i,N-3} - \alpha_1 \delta_{i,N-2} + (\alpha_1^2 - \alpha_2)\delta_{i,N-1}] t_4 \qquad i = 0, 1, \ldots, N-1. \tag{33}$$

Equation (29), together with equations (30) and (33), leads to the Jacobi inversion problem for the soliton equation (7) with $n = 4$:

$$\frac{1}{2}\sum_{k=1}^{N}\int^{u_k} \frac{\lambda^i}{\sqrt{P(\lambda)K(\lambda)}}\,\mathrm{d}\lambda = c_i + [\delta_{i,N-3} - \alpha_1\delta_{i,N-2}]t_4 - [x - (\alpha_1^2 - \alpha_2)t_4]\delta_{i,N-1} \qquad i = 0, 1, \ldots, N-1. \tag{34}$$

By using standard Jacobi inversion techniques [16], the solution (q, r) of the soliton equation (7), which are the symmetric functions of $u_k, k = 1, \ldots, N$ defined by (24), can be given an explicit form in terms of Riemann theta functions.

The method presented above can be applied to all factorizations of (11) and (12) for equations in the JM hierarchy and for other soliton hierarchies.

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